Directed polymer in random environment and last passage percolation*

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Abstract

The sequence of random probability measures ν_n that gives a path of length n, $\frac{1}{n}$ times the sum of the random weights collected along the paths, is shown to satisfy a large deviations principle with good rate function the Legendre transform of the free energy of the associated directed polymer in a random environment.

Consequences on the asymptotics of the typical number of paths whose collected weight is above a fixed proportion are then drawn.

Keywords: directed polymer, random environment, partition function, last passage percolation.
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1 Introduction

Last passage percolation

To each site (k,x) of $\mathbb{N} \times \mathbb{Z}^d$ is assigned a random weight $\eta(k,x)$. The $(\eta(k,x))_{k\geq 1, x\in\mathbb{Z}^d}$ are taken IID under the probability measure $\mathbf{Q}$.
The set of oriented paths of length n starting from the origin is

$$\Omega_n = \Big\{\omega = (\omega_0, \dots, \omega_n) : \omega_i \in \mathbb{Z}^d, \omega_0 = 0, |\omega_i - \omega_{i-1}| = 1\Big\}.$$

The weight (energy, reward) of a path is the sum of weights of visited sites:

$$H_n = H_n(\omega, \eta) = \sum_{k=1}^{n} \eta(k, \omega_k) \quad (n \geq 1, \omega \in \Omega_n).$$

Observe that when $\eta(k,x)$ are Bernoulli(p) distributed

$$\mathbf{Q}(\eta(k,x) = 1) = 1 - \mathbf{Q}(\eta(k,x) = 0) = p \in (0,1)\,,$$

the quantity $\frac{H_n}{n}(\omega,\eta)$ is the proportion of *open* sites visited by ω, and it is natural to consider for $0 < \rho < 1$,

$$N_n(\rho) = \text{number of paths of length } n \text{ such that } H_n(\omega,\eta) \geq n\rho.$$

The problem of ρ-percolation, as we learnt it from Comets, Popov and Vachkovskaia [8] and Kesten and Sidoravicius [12], is to study the behaviour of $N_n(\rho)$ for large n and different values of ρ.

Directed polymer in a random environment

We are going to consider fairly general environment distributions, by requring first that they have exponential moments of any order:

$$\lambda(\beta) = \log \mathbf{Q}\Big(e^{\beta\eta(k,x)}\Big) < +\infty \quad (\beta \in \mathbb{R})\,,$$

and second that they satisfy a logarithmic Sobolev inequality (see e.g. [2]): in particular we can apply our result to bounded support and Gaussian environments.
The polymer measure is the random probability measure defined on the set of oriented paths of length n by:

$$\mu_n(\omega) = (2d)^{-n}\, \frac{e^{\beta H_n(\omega,\eta)}}{Z_n(\beta)} \qquad (\omega \in \Omega_n),\,,$$

with $Z_n(\beta)$ the partition function

$$Z_n(\beta) = Z_n(\beta,\eta) = (2d)^{-n} \sum_{\omega\in\Omega_n} e^{\beta H_n(\omega,\eta)} = \mathbf{P}\left(e^{\beta H_n(\omega,\eta)}\right),$$

where $\mathbf{P}$ is the law of simple random walk on $\mathbb{Z}^d$ starting from the origin. Bolthausen [3] proved the existence of a deteministic limiting free energy

$$p(\beta) = \lim_{n\to+\infty} \frac{1}{n}\mathbf{Q}(\log Z_n(\beta)) = \mathbf{Q}\, a.s. \lim_{n\to+\infty} \frac{1}{n} \log Z_n(\beta)\,.$$

Thanks to Jensen's inequality, we have the upper bound $p(\beta) \leq \lambda(\beta)$ and it is conjectured (and partially proved, see [7, 6]) that the behaviour of a typical path under the polymer measure is diffusive iff $\beta \in \mathcal{C}_\eta$ the critical region

$$\mathcal{C}_\eta = \{\beta\in\mathbb{R} : p(\beta) = \lambda(\beta)\}\,.$$

In dimension $d = 1$, $\mathcal{C}_\eta = \{0\}$ and in dimensions $d \geq 3$, $\mathcal{C}_\eta$ contains a neighborhood of the origin (see [3, 9]).

The main theorem

The connection between Last passage percolation and Directed polymer in random environment is made by the family $(\nu_n)_{n\in\mathbb{N}}$ of random probability measures on the real line:

$$\nu_n(A) = \frac{1}{|\Omega_n|} \sum_{\omega\in\Omega_n} \mathbf{1}_{\left(\frac{H_n}{n}(\omega,\eta)\in A\right)} = \mathbf{P}\left(\frac{H_n}{n}(\omega,\eta) \in A\right).$$

Indeed,

$$N_n(\rho) = \sum_{\omega\in\Omega_n} \mathbf{1}_{(H_n(\omega,\eta)\geq n\rho)} = (2d)^n \nu_n([\rho,+\infty))\,.$$

The main result of the paper is

Theorem 1. $\mathbf{Q}$ *almost surely, the family* $(\nu_n)_{n\in\mathbb{N}}$ *satisfies a large deviations principle with good rate function* $I = p^*$ *the Legendre transform of the free energy of the directed polymer.*

Let $m = \mathbf{Q}(\eta(k,x))$ be the average weight of a path $m = \mathbf{Q}\big(\frac{H_n}{n}(\omega,\eta)\big)$. It is natural to consider the quantities:

$$N_n(\rho) = \begin{cases} \sum_{\omega\in\Omega_n} \mathbf{1}_{(H_n(\omega,\eta)\geq n\rho)} & \text{if } \rho \geq m\,, \\ \sum_{\omega\in\Omega_n} \mathbf{1}_{(H_n(\omega,\eta)\leq n\rho)} & \text{if } \rho < m\,. \end{cases}$$

A simple exchange of limits $\beta \to \pm\infty$, and $n \to +\infty$, yields the following

$$\rho^{\pm} = \mathbf{Q}\, a.s. \lim_{n\to+\infty} \max_{\omega\in\Omega_n} \pm \frac{H_n}{n}(\omega,\eta) = \lim_{\beta\to+\infty} \frac{p(\pm\beta)}{\beta} \in [0,+\infty]\,.$$

Repeating the proof of Theorem 1.1 of [8] gives

Corollary 2. *For $-\rho^- < \rho < \rho^+$, we have $\mathbf{Q}$ almost surely,*

$$\lim_{n\to+\infty} (N_n(\rho))^{\frac{1}{n}} = (2d)e^{-I(\rho)}\,.$$

We can then translate our knowledge of the critical region $\mathcal{C}_\eta$, into the following remark. Let

$$\mathcal{V}_\eta = \{\rho \in \mathbb{R} : I(\rho) = \lambda^*(\rho)\}\,.$$

In dimension $d = 1$, $\mathcal{V}_\eta = \{m\}$ and in dimensions $d \geq 3$, $\mathcal{V}_\eta$ contains a neighbourhood of m.
This means that in dimensions $d \geq 3$, the typical large deviation of $\frac{H_n}{n}(\omega,\eta)$ close to its mean is the same as the large deviation of $\frac{1}{n}(\eta_1 + \cdots + \eta_n)$ close to its mean, with η_i IID. There is no influence of the path ω : this gives another justification to the name weak-disorder region given to the critical set $\mathcal{C}_\eta$.

2 Proof of the main theorem

Observe that for any $\beta \in \mathbb{R}$ we have:

$$\int e^{\beta n x} d\nu_n(x) = \mathbf{P}\left(e^{\beta H_n(\omega,\eta)}\right) = Z_n(\beta) \quad \mathbf{Q}\, a.s.. \tag{1}$$

Consequently, since $e^u + e^{-u} \geq e^{|u|}$, we obtain for any $\beta > 0$,

$$\limsup_{n\to+\infty} \frac{1}{n} \log\left(\int e^{\beta n|x|} d\nu_n(x)\right) \leq p(\beta) + p(-\beta) < +\infty\,,$$

and the family $(\nu_n)_{n\geq 0}$ is exponentially tight (see Dembo and Zeitouni[10], or Feng and Kurtz[11]). We only need to show now that for a lower semi-continuous function I, and for $x \in \mathbb{R}$

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \log \nu_n((x-\delta, x+\delta)) = I(x)\,, \tag{2}$$

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \nu_n([x-\delta, x+\delta]) = I(x). \tag{3}$$

From these, we shall infer that $(\nu_n)_{n\in\mathbb{N}}$ follows a large deviations principle with good rate function I. Eventually, equation (1) and

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n(\beta) = p(\beta)$$

will imply, by Varadhan's lemma that I and p are Legendre conjuguate:

$$I(x) = p^*(x) = \sup_{\beta\in\mathbb{R}} (x\beta - p(\beta))\,.$$

The strategy of proof finds its origin in Varadahan's seminal paper[13], and has already succesfully been applied in [5]. Let us define for $\lambda > 0, x \in \mathbb{Z}, a \in \mathbb{R}$

$$V_n^{(\lambda)}(x, a; \eta) = \log \mathbf{P}^x \left(e^{-\lambda |H_n(\omega,\eta) - a|} \right) = V^{(\lambda)}(0, a; \tau_{o,x} \circ \eta)\,,$$

with $\tau_{k,x}$ the translation operator on the environment defined by :

$$\tau_{k,x} \circ \eta(i, y) = \eta(k + i, x + y)\,,$$

and $\mathbf{P}^x$ the law of simple random walk starting from x.

Step 1 The functions $v_n^{(\lambda)}(a) = \mathbf{Q}\big(V^{(\lambda)}(0, a; \eta)\big)$ satisfy the inequality

$$v_{n+m}^{(\lambda)}(a + b) \geq v_n^{(\lambda)}(a) + v_m^{(\lambda)}(b) \qquad (n, m \in \mathbb{N};\ a, b \in \mathbb{R})\,. \tag{4}$$

Proof. Since $|H_{n+m} - (a + b)| \leq |H_n - b| + |(H_{n+m} - H_n) - a|$ we have

$$\begin{aligned}
V_{n+m}^{(\lambda)}(x, a; \eta) &\geq \log \mathbf{P}^x \left(e^{-\lambda|H_n - b|} e^{-\lambda|(H_{n+m} - H_n) - a|} \right) \\
&= \log \mathbf{P}^x \left(e^{-\lambda|H_n - b|} e^{V_m^{(\lambda)}(0,a;\tau_{n,S_n}\circ\eta)} \right) \\
&= \log \sum_y \mathbf{P}^x \left(e^{-\lambda|H_n - b|}\, \mathbf{1}_{(S_n = y)} \right) e^{V_m^{(\lambda)}(0,a;\tau_{n,y}\circ\eta)} \\
&= V_n^{(\lambda)}(x, b; \eta) + \log \left(\sum_y \sigma_n(y) e^{V_m^{(\lambda)}(0,a;\tau_{n,y}\circ\eta)} \right) \\
&\geq V_n^{(\lambda)}(x, b; \eta) + \sum_y \sigma_n(y) V_m^{(\lambda)}(0, a; \tau_{n,y} \circ \eta) \qquad \text{(Jensen's inequality)}\,,
\end{aligned}$$

with σ_n the probability measure on $\mathbb{Z}^d$:

$$\sigma_n(y) = \frac{1}{V_n^{(\lambda)}(x, b; \eta)} \mathbf{P}^x \left(e^{-\lambda|H_n - b|}\, \mathbf{1}_{(S_n = y)} \right) \quad (y \in \mathbb{Z}^d)\,.$$

Observe that the random variables $\sigma_n(y)$ are measurable with respect to the sigma field $\mathcal{G}_n = \sigma(\eta(i,x) : i \le n, x \in \mathbb{Z}^d)$, whereas the random variables $V_m^{(\lambda)}(0,a;\tau_{n,y} \circ \eta)$ are independent from $\mathcal{G}_n$. Hence, by stationarity,

$$
\begin{aligned}
v_{n+m}^{(\lambda)}(x,a;\eta) &= \mathbf{Q}\Big(V_{n+m}^{(\lambda)}(x,a;\eta)\Big) \\
&\ge v_n^{(\lambda)}(b) + \sum_y \mathbf{Q}(\sigma_n(y))\mathbf{Q}\Big(V_m^{(\lambda)}(0,a;\tau_{n,y}\circ\eta)\Big) \\
&= v_n^{(\lambda)}(b) + \sum_y \mathbf{Q}(\sigma_n(y)) v_m^{(\lambda)}(a) \\
&= v_n^{(\lambda)}(b) + v_m^{(\lambda)}(a)\mathbf{Q}\left(\sum_y \sigma_n(y)\right) \\
&= v_n^{(\lambda)}(b) + v_m^{(\lambda)}(a)\,.
\end{aligned}
$$

□

Step 2 There exists a function $I^{(\lambda)} : \mathbb{R} \to \mathbb{R}^+$ convex, non negative, Lipschitz with constant λ, such that

$$
-\lim_{n\to\infty} \frac{1}{n} v_n^{(\lambda)}(a_n) = I^{(\lambda)}(\xi) \qquad (\text{if } \frac{a_n}{n} \to \xi \in \mathbb{R}))\,. \tag{5}
$$

Proof. This is a standard subadditivity argument (see e.g. Varadhan [13] or Alexander [1]) combined with the Lipschitz property of $V^{(\lambda)}$: from $|H_n - a| \le |H_n - b| + |a-b|$ we infer that

$$
V_n^{(\lambda)}(0,a;\eta) \ge V_n^{(\lambda)}(0,a;\eta) + \lambda|a-b|\,.
$$

□

Step 3 $\mathbf{Q}$ almost surely, for any $\xi \in \mathbb{R}$,

$$
\lim_{n\to\infty} -\frac{1}{n} \log \mathbf{P}\left(e^{-\lambda|H_n - a_n|}\right) = I^{(\lambda)}(\xi)\,. \tag{6}
$$

Proof. Since the functions are Lipschitz, it is enough to prove that for any fixed $\xi \in \mathbb{Q}$, (6) holds a.s. This is where we use the restrictive assumptions made on the distribution of the environment. If the distribution of η is with bounded support, or Gaussian, or more generally satisfies a logarithmic Sobolev inequality, then it has the gaussian concentration of measure

property (see[2]): for any 1-Lipschitz function F of independent random variables distributed as η,

$$\mathbf{P}\left(|F-\mathbf{P}\left(F\right)|\geq r\right)\leq 2e^{-r^2/2}\quad (r>0).$$

It is easy to prove, as in Proposition 1.4 of [4], that the function

$$(\eta(k,x),k\leq n,|x|\leq n)\rightarrow \log \mathbf{P}\left(e^{-\lambda|H_n(\omega,\eta)-a|}\right)$$

is Lipschitz, with respect to the euclidean norm, with Lipschitz constant at most $\lambda\sqrt{n}$. Therefore, the Gaussian concentration of measure yields

$$\mathbf{Q}\left(\left|V_n^{(\lambda)}(0,a;\eta)-v_n^{(\lambda)}(a)\right|\geq u\right)\leq 2e^{-\frac{\lambda^2u^2}{2n}}\,.$$

We conclude by a Borel Cantelli argument combined with (5)

$\square$

Observe that for fixed $\xi\in\mathbb{R}$, the function $\lambda\rightarrow I^{(\lambda)}(\xi)$ is increasing ; we shall consider the limit:

$$I(\xi)=\lim_{\lambda\uparrow+\infty}\uparrow I^{(\lambda)}(\xi)$$

which is by construction non negative, convex and lower semi continuous.

Step 4 The function I satisfy (2) and (3).

Proof. Given, $\xi\in\mathbb{R}$ and $\lambda>0,\delta>0$, we have

$$\mathbf{P}\left(\left|\frac{H_n}{n}(\omega,\eta)-\xi\right|\leq\delta\right)=\mathbf{P}\left(e^{-\lambda n\left|\frac{H_n}{n}(\omega,\eta)-\xi\right|}\geq e^{-\lambda n\delta}\right)\leq e^{\lambda n\delta}\mathbf{P}\left(e^{-\lambda|H_n-n\xi|}\right).$$

Therefore,

$$\limsup\frac{1}{n}\log\nu_n([\xi-\delta,\xi+\delta])\leq\lambda\delta-I^{(\lambda)}(\xi)$$
$$\limsup_{\delta\to0}\limsup\frac{1}{n}\log\nu_n([\xi-\delta,\xi+\delta])\leq -I^{(\lambda)}(\xi)$$

and we obtain by letting $\lambda\rightarrow+\infty$,

$$\limsup_{\delta\to0}\limsup\frac{1}{n}\log\nu_n([\xi-\delta,\xi+\delta])\leq -I(\xi)\,.$$

Given $\xi \in \mathbb{R}$ such that $I(\xi) < +\infty$, and $\delta > 0$, we have for $\lambda > 0$,

$$\mathbf{P}\left(\left|\frac{H_n}{n} - \xi\right| < \delta\right) \geq \mathbf{P}\left(e^{-\lambda|H_n - n\xi|}\right) - e^{-\lambda\delta n}\,.$$

Hence, if we choose $\lambda > 0$ large enough such that $\lambda\delta > I(\xi) \geq I^{(\lambda)}(\xi)$, we obtain

$$\liminf_{n\to+\infty} \frac{1}{n}\log\nu_n((\xi-\delta,\xi+\delta)) \geq -I^{(\lambda)}(\xi) \geq -I(\xi)$$

and therefore

$$\liminf_{\delta\to 0}\ \liminf_{n\to+\infty} \frac{1}{n}\log\nu_n((\xi-\delta,\xi+\delta)) \geq -I(\xi)\,.$$

□

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